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DISCUSSION OF A METHOD FOR FINDING NUMERICAL SQUARE ROOTS

By C. L. Bouton

In a recent article* Professor E. R. Hedrick gave a method for approximating to the positive square root of any positive number. Inasmuch as he merely states the method without giving proofs, and as the proof itself is a rather pretty exercise in the elementary theory of limits, it seems worth while to give a discussion of the method. The facts that the method is really Newton's method of approximating to the square root, and that it may easily be extended to the p^{th} root, are of interest. Professor Hedrick's plea that this method be substituted for the method of extracting square root ordinarily used seems to be scarcely justified on the ground of rapidity of convergence, although there may be pedagogical advantages in its use with young pupils. The method has the further disadvantage that when applied to a number which is a perfect square, it yields the square root only as the result of an infinite process, unless indeed the *first* trial root should turn out to be the exact root; whereas, the ordinary process applied to this case always terminates in a finite number of steps, provided that the given number is an integer, or a finite decimal fraction.

The process is the following: Let a be the given number, select any positive number n, and form in succession

$$(1)\qquad \left\{\begin{aligned} n_1 &= \frac{1}{2}\left(n + \frac{a}{n}\right), \\ n_2 &= \frac{1}{2}\left(n_1 + \frac{a}{n_1}\right), \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ n_i &= \frac{1}{2}\left(n_{i-1} + \frac{a}{n_{i-1}}\right). \end{aligned}\right.$$

Then we shall prove that the numbers of the sequence $n_1, n_2, \cdots, n_i$, form a decreasing set, and approach $\sqrt{a}$ as their limit. The process converges rapidly enough to be useful in computing numerical square roots. For example, let $a = 2$, and choose $n = 1$. Then we compute in succession

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* *Western Journal of Education*, vol. I, October, 1908, p. 252.

$$n_1 = 1.5, \quad n_2 = 1.417, \quad n_3 = 1.414216$$

Thus in three steps the error is about 2 in 10^6.

To prove that $\lim\limits_{i=\infty} (n_i) = \sqrt{a}$, and to estimate the error at any step, let us proceed as follows: First we have the identity

$$(2) \qquad n_i^2 - a = \frac{1}{4}\left(n_{i-1} + \frac{a}{n_{i-1}}\right)^2 - a = \left(\frac{n_{i-1}^2 - a}{2n_{i-1}}\right)^2 \geqq 0,$$

whence

$$(3) \qquad n_i \geqq \sqrt{a} \qquad (i = 1, 2, \cdots).$$

That is, $n_1, n_2, \cdots$ are all greater than $\sqrt{a}$, unless n was chosen equal to $\sqrt{a}$, in which case $n_1 = n_2 = \cdots = \sqrt{a}$. From this point on we shall assume that n was not chosen equal to $\sqrt{a}$, so that in (2) and (3) the equality sign cannot hold. Moreover, from (1) and (3) we have

$$(4) \qquad n_{i-1} - n_i = \frac{n_{i-1}^2 - a}{2n_{i-1}} > 0 \qquad (i = 2, 3, \cdots),$$

so that

$$(5) \qquad n_i < n_{i-1} \qquad (i = 2, 3, \cdots).$$

Therefore the variable n_i always decreases when i increases, but n_i is always greater than $\sqrt{a}$. Hence by a fundamental principle in the theory of limits* the limit of n_i exists when $i = \infty$; let us call it N. We readily see that this limit is $\sqrt{a}$, for from (2) and (4) we have

$$(6) \quad n_i - \sqrt{a} = \frac{1}{n_i + \sqrt{a}}\left(\frac{n_{i-1}^2 - a}{2n_{i-1}}\right)^2 = \frac{(n_{i-1} - n_i)^2}{n_i + \sqrt{a}} \qquad (i = 1, 2, \cdots).$$

Passing to the limit for $i = \infty$, we get

$$N - \sqrt{a} = \frac{(N - N)^2}{N + \sqrt{a}} = 0,$$

$$\therefore \quad N = \sqrt{a}.$$

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* See, for example, Osgood, *Introduction to Infinite Series*, p. 4.

This proves the convergence of the process. It is easy to set limits to the error at any step, and to form some idea of the rapidity of the convergence. We have from (6) and (3)

$$(7) \qquad \frac{(n_{i-1}-n_i)^2}{2n_i} < n_i - \sqrt{a} < \frac{(n_{i-1}-n_i)^2}{2\sqrt{a}} < \frac{n_i(n_{i-1}-n_i)^2}{2a}.$$

That is, if we set $n_{i-1} - n_i = \delta_i$ and $n_i - \sqrt{a} = \epsilon_i$, then

$$(8) \qquad \frac{\delta_i^2}{2n_i} < \epsilon_i < \frac{n_i\delta_i^2}{2a} \qquad (i = 1, 2, \cdots).$$

Furthermore, from (7) and (3)

$$(9) \qquad \frac{n_i - \sqrt{a}}{\sqrt{a}} < \frac{(n_{i-1}-n_i)^2}{2a} < \frac{1}{2}\left(\frac{n_{i-1}-\sqrt{a}}{\sqrt{a}}\right)^2.$$

Let $\dfrac{n_i - \sqrt{a}}{\sqrt{a}} = r_i$, so that r_i is the relative or percentage error at the i^{th} step. Then

$$(10) \qquad r_i < \tfrac{1}{2} r_{i-1}^2 \qquad (i = 2, 3, \cdots).$$

Repeated application of (10) shows that

$$(11) \qquad r_i < \frac{1}{2^{2^{i+1}-1}}\left[\frac{(n^2-a)^2}{n^2 a}\right]^{2^{i-1}} \qquad (i = 1, 2, \cdots).$$

Formula (8) shows that the actual error of any approximation is nearly proportional to the square of the difference between that approximation and the preceding one. Formula (10) gives the relation between the *relative* error at any step, and that of the preceding step. It shows that when working decimally the number of significant figures retained at any step should be double the number which are correct at the preceding step. These formulae both show that while the process may not converge very rapidly at first, it later converges with extreme rapidity. Thus in the computation for $\sqrt{2}$ we found $r_3 < 2 \times 10^{-6}$, hence $r_4 < 2 \times 10^{-12}$ and $r_5 < 2 \times 10^{-24}$.

Formula (11) enables us to estimate the number of steps necessary to obtain a given degree of approximation, and furthermore shows the effect of the choice of the initial n. Of course the nearer n is chosen to $\sqrt{a}$ the more rapid is the convergence. The method is therefore particularly useful for computing square roots when an approximate value is known, as for example, in extending a given table of square roots to a larger number of places. Thus, if we take $a = 3$, and choose $n = 1.732$, a single step gives $n_1 = 1.732050808314$. If to this we apply the correction computed from (8) we find 1.732050807569, while the value of $\sqrt{3}$ correct to thirteen decimal places is 1.7320508075689.

Furthermore, it should be noticed that the commission of an arithmetical error at *any* stage of the work does not vitiate the ultimate result, for it simply amounts to starting over with a new value of n. In fact, such a mistake may even *increase* the rapidity of the convergence.

A geometrical proof of the convergency of the process is easily given. Draw the rectangular hyperbola whose equation is

$$xy = a.$$

On this hyperbola connect the points $\left(n, \frac{a}{n}\right)$ and $\left(\frac{a}{n}, n\right)$. Let C_1 be the middle point of this chord. It is easy to see that the coordinates of C_1 are (n_1, n_1). Then repeat the process with n_1 in place of n, and so on indefinitely. An examination of the figure shows at once that the points C_1, C_2, $C_3 \cdots$ are moving down the diagonal towards the vertex, but that they never go beyond the vertex. Hence the limit of n_i is $\sqrt{a}$. This figure further shows that

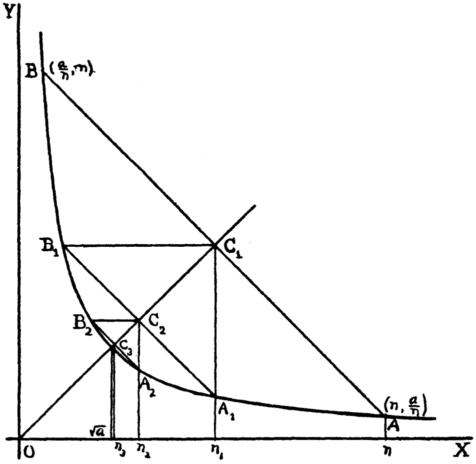

FIG. 1.

the value of n assumed at the start has no influence on the final result, but that if a value of n near $\sqrt{a}$ be chosen the process will converge much more rapidly than if we begin with a value remote from $\sqrt{a}$.

It is interesting to apply this geometrical method to trying to find the square root of a negative number. The process does not converge.

Finally, it should be noticed that this process is exactly that given by Newton's method for approximating to the positive root of the equation

$$x^2 - a = 0.$$

In that method, the curve

$$y = x^2 - a$$

is drawn, any value of x is selected, the tangent is drawn at the corresponding point of the curve, and its intersection with the X-axis gives a new value of x, with which the process is repeated. In this particular case the equation of the tangent at $(n, n^2 - a)$ is

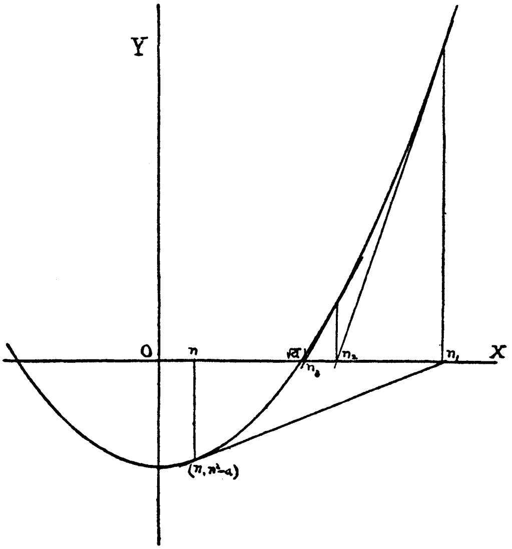

FIG. 2.

$$y - (n^2 - a) = 2n(x - n)$$

or

$$2nx = y + n^2 + a$$

whence

$$n_1 = \frac{1}{2}\left(n + \frac{a}{n}\right), \text{ if } n_1 \text{ be the } X\text{-intercept;}$$

and the process is repeated. But this is exactly the method described above.

Newton's method applied to the p^{th} root gives at once

$$(12) \qquad \left\{ \begin{array}{l} n_1 = \dfrac{1}{p}\left[(p-1)n + \dfrac{a}{n^{p-1}}\right], \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ n_i = \dfrac{1}{p}\left[(p-1)n_{i-1} + \dfrac{a}{n_{i-1}^{p-1}}\right]. \end{array} \right.$$

If we write

$$n_{i-1} - n_i = \delta_i, \quad n_i - \sqrt[p]{a} = \epsilon_i, \quad \frac{\epsilon_i}{\sqrt[p]{a}} = r_i$$

it is easy to prove from (12) that

$$\frac{(p-1)\delta_i^2}{2n_i} < \epsilon_i < \frac{(p-1)\delta^2}{2[n_i - (p-1)\delta_i]},$$

and

$$r_i < \frac{p-1}{2} r_{i-1}^2 (1 + r_{i-1})^{p-2}.$$

These formulae are direct generalizations of (8) and (10), for the actual and relative error of any approximation.

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